



# Giant Magnons for SYM and ABJM Theories

Michael Abbott 

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Describing work with Inês Aniceto  and Olof Olhsson Sax   
in arxiv:0811.2423 and 0903.3365,

and work scooped by Hatsuda & Tanaka , 0910.5315,

and work in progress with Inês Aniceto  and Diego Bombardelli .

## Outline

1. Strings and spin chains in AdS/CFT
2. ABJM and  $AdS_4 \times CP^3$
3. Classification of giant magnons in  $CP^3$
4. Finite-size corrections:  $J < \infty$
5. Worldsheet scattering
6. Quantum corrections:  $\lambda < \infty$

# Spinning Strings, Solitons, and Spin Chains

Limit of AdS/CFT:

- Strictly planar:  $N \rightarrow \infty$ .
- Large R-charge  $J \rightarrow \infty$  (Later expand in  $J/\sqrt{\lambda} \gg 1$ .)
- But  $\lambda$  independent of this (unlike BMN's  $\lambda \sim J^2$ )
- Finite  $p$  (unlike BMN's  $p \sim 1/J$ )

We have perturbative gauge theory at  $\lambda \ll 1$ ,  
semiclassical strings at  $\lambda \gg 1$ ,  
and a gap in the middle.

Some success at bridging this gap...

(Finite  $p$  corresponds to exploring more of the geometry than the PP-wave.)



## Spin Chain Picture [Minahan & Zarembo, 2002]

Start with:

- $su(2)$  sector (fields  $Z = \Phi_1 + i\Phi_2$  and  $X = \Phi_3 + i\Phi_4$  only)
- at one loop (order  $\lambda$ ).

The simplest large- $J$  operator is

$$O_{\text{vac}} = \text{Tr}(\underbrace{ZZZZ}_{\text{ZZZZ}} \dots Z) \quad \text{with } \Delta = J$$

and we consider operators with ‘impurities’  $X$  added.

The dilatation operator is  $\Delta = \Delta_0 + \Delta_1 + \dots$ , and it is easy to show that

$$\Delta_1 = \frac{\lambda}{8\pi^2} \sum_i (1_{i,i+1} - P_{i,i+1}) = \frac{\lambda}{8\pi^2} \sum_i \mathcal{H}_{i,i+1}$$

which is exactly the Heisneberg  $XXX-\frac{1}{2}$  spin chain Hamiltonian.

This is the system [Bethe, 1931] solved exactly: construct the full spectrum using two-particle S-matrix. Perhaps the simplest integrable system...



A single excitation is a magnon

$$O = \sum_x e^{ipx} \text{Tr} \left( \underset{x}{ZZ} \underset{x}{X} \underset{x}{ZZ} \dots Z \right)$$

Always understood to be part of a set with  $\sum_i p_i = 0$ .

More formally,  $p \neq 0$  gives some central generator  $K$  nonzero eigenvalue.

In the slightly larger  $su(2, 3)$  sector, algebra acting on impurities is  $su(2|2) \times u(1)^2$ , and this alone fixes dispersion relation [Beisert, 2006]

$$E(p) \equiv \Delta - J = \sqrt{Q^2 + f(\lambda)^2 \sin^2 \frac{p}{2}}$$

for a bound state of  $Q$  magnons. 'Experiments' fix  $f(\lambda) = \sqrt{\lambda}/\pi$ .

Hybrid system:

$$\begin{aligned} E &= p^2/2m && \text{non-relativistic} \\ &= \sin(p/2) && \text{on a lattice} \\ &= \sqrt{m^2 + p^2} && \text{relativistic} \end{aligned}$$

## String Solitons [Hofman & Maldacena, 2006]

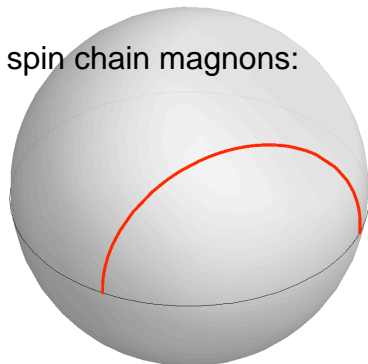
Giant magnons are classical string solutions dual to spin chain magnons:

$$X^1 + iX^2 = e^{it} \left( \cos \frac{p}{2} + i \sin \frac{p}{2} \tanh(u) \right)$$

$$X^3 = \sin \frac{p}{2} \operatorname{sech}(u)$$

where  $u = (x - t \cos \frac{p}{2}) / \sin \frac{p}{2}$ . Charges

$$E = \Delta - J = \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2}$$



- As  $x \rightarrow \pm\infty$ , approach cusps  $\delta\phi = p$  apart. Set with  $\sum_i p = 0$ .
- ‘Solitons’ because, under the Pohlmeyer map to (classical) sine-gordon theory, these become simple kinks.
- Generalise some of **GKP**’s spinning strings to general  $p$ , allowing nonzero worldsheet velocity  $v = \cos(p/2)$ , allowing scattering.
- No Lorentz invariance: conformal gauge with  $X^0 = t$ .

## Dyonic Magnons [Dorey] [Chen, Dorey, Okamura] 2006

Turn on 2nd charge  $Q \sim \sqrt{\lambda}$ , leading to dispersion relation

$$E = \Delta - J = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$$

Explicitly,

$$X^1 + iX^2 = e^{it} \left( \cos \frac{p}{2} + i \sin \frac{p}{2} \tanh U \right)$$

$$X^3 + iX^4 = e^{iV} \sin \frac{p}{2} \operatorname{sech} U$$

where

$$U = (x \cosh \beta - t \sinh \beta) \cos \alpha \quad \cot \alpha = \frac{2r}{1-r^2} \sin \frac{p}{2}$$

$$V = (t \cosh \beta - x \sinh \beta) \sin \alpha \quad \tanh \beta = \frac{2r}{1+r^2} \cos \frac{p}{2}.$$

- Can view these as allowing orientation vector of basic magnon to rotate in 3-4 plane. All orientations are equivalent.

# The New Example: ABJM

# 2

Instead of six scalars  $\Phi_i$  in the adjoint of  $U(N)$ , we now have four scalars  $Y_i$  in  $(N, \bar{N})$  of  $U(N) \times U(N)$ .

Gauge-invariant long operators alternate  $(N, \bar{N})$  and  $(\bar{N}, N)$ :

$$O = \chi_{abc\dots}^{xyz\dots} \text{Tr} \left( Y^a Y_x^\dagger Y^b Y_y^\dagger Y^c Y_z^\dagger \dots \right)$$

Call this the vacuum:

$$O_{vac} = \text{Tr} \left( Y^1 Y_4^\dagger \right)^J$$

In the  $su(2) \times su(2)$  sector at two loops the dilatation operator is

$$\Delta = \frac{J}{2} + \frac{\lambda^2}{4} \left( \sum_{i \text{ even}} \mathcal{H}_{i,i+2} + \sum_{i \text{ odd}} \mathcal{H}_{i,i+2} \right) + \dots$$

i.e. two decoupled XXX spin chains [Minahan & Zarembo] [Bak & Rey]

Same extended  $su(2|2)^2$  symmetry constrains the dispersion relation to be

$$E = \Delta - \frac{J}{2} = \sqrt{\frac{1}{4} + 4 h^2(\lambda) \sin^2 \frac{\rho}{2}}$$

but  $h$  is no longer constant:

$$h^2(\lambda) = \begin{cases} \frac{\lambda}{2} + c\sqrt{\frac{\lambda}{2}} + \mathcal{O}(\lambda^0) & \lambda \gg 1 \\ \lambda^2 + h_4\lambda^4 + \mathcal{O}(\lambda^6) & \lambda \ll 1 \end{cases}$$

Subleading terms:

- $h_4 = -16 + 4\zeta(2) \approx -9.42$ , according to [Minahan, Ohlsson Sax, Sieg] (4-loop gauge theory calculation)
- Perhaps  $c = -\frac{\log 2}{2\pi} \approx -0.11$ , say [McLoughlin, Roiban, Tseytlin], but perhaps  $c = 0\dots$  [Gromov & Mikhaylov] (discuss later)

These rule out simple interpolating functions.

# Giant Magnons in $CP^3$



In 't Hooft limit, IIA strings in  $AdS_4 \times CP^3$ .

Expect similar solutions to those in  $S^5$  case, including giant magnons.

Classification problem, a year ago:

- several giant magnons known in the string  $\sigma$ -model
- and several in the algebraic curve too, but not the same ones.

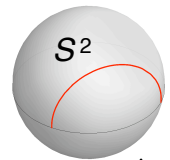
Known  $\sigma$ -model solutions were original solution in  $S^2$ -like subspaces:

- $CP^1$ , with  $E = \sqrt{2\lambda} \sin \frac{p}{2}$  (closed when  $\sum_i p_i = 0, 2\pi, \dots$ )
- $RP^2$ , with  $E = 2\sqrt{2\lambda} \sin \frac{p'}{2}$  (closed when  $\sum_i p'_i = 0, \pi, \dots$ )

and an embedding of Dorey's generalising the latter:

- $RP^3$ , with  $E = \sqrt{\frac{Q^2}{4} + 8\lambda \sin^2 \frac{p'}{2}}$

**S<sup>5</sup> solutions:**



[Hofman & Maldacena]

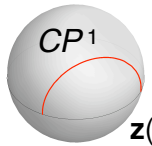


[Dorey, 2006]

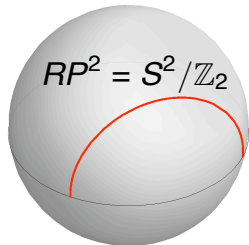
There are various tempting subspaces including  $S^2 \times S^2$ , which the equations of motion don't respect.

[MCA & Aniceto, 2008]

**CP<sup>3</sup> solutions:**



New?



**Embedding into C<sup>4</sup>:**

$$\mathbf{z}(x, t) = \begin{pmatrix} e^{\frac{i}{2}\phi_{\text{mag}}(2x, 2t)} \sin\left(\frac{1}{2}\theta_{\text{mag}}(2x, 2t)\right) \\ 0 \\ 0 \\ e^{-\frac{i}{2}\phi_{\text{mag}}(2x, 2t)} \cos\left(\frac{1}{2}\theta_{\text{mag}}(2x, 2t)\right) \end{pmatrix}$$

[Gaiotto, Giombi, Yin]  
[Grignani, Harmark, Orselli]

$\mathbf{z}(x, t) = \dots$

$$\mathbf{z}(x, t) = \begin{pmatrix} e^{i\phi_{\text{dyon}}(x, t)} \sin\theta_{\text{dyon}}(x, t) \\ e^{iV} \cos\theta_{\text{dyon}}(x, t) \\ e^{-iV} \cos\theta_{\text{dyon}}(x, t) \\ e^{-i\phi_{\text{dyon}}(x, t)} \sin\theta_{\text{dyon}}(x, t) \end{pmatrix}$$

[Ahn, Bozhilov, Rashkov] [Ryang]

## Constructing $CP^2$ Solution

The  $CP^1$  solution has no orientation, so we deform to  $\xi \neq \pi/2$  to give it one:

$$\mathbf{z} = \begin{pmatrix} \sin \xi \cos(\vartheta_2/2) e^{i\varphi_2/2} \\ \cos \xi e^{i\varphi_1/2} \\ 0 \\ \sin \xi \sin(\vartheta_2/2) e^{-i\varphi_2/2} \end{pmatrix}$$

with ansatz

$$\begin{aligned} \varphi_2 &= 2t, & \varphi_1 &= -2\omega t, \\ \cos \vartheta_2 &= \operatorname{sech} \left( \sqrt{1 - \omega^2} 2x \right), & \xi &= \frac{\pi}{2} - e(x). \end{aligned}$$

for  $p = \pi$  case.

We were able to solve for  $e(x)$ : [MCA, Aniceto, Ohlsson Sax, 2009]

$$\cos^2(e(x)) = \frac{1}{1 + \omega \operatorname{sech} \left( \sqrt{1 - \omega^2} 2x \right)}$$

It has charges

$$E = \Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 2\lambda} \xrightarrow{p \neq \pi} \sqrt{\frac{Q^2}{4} + 2\lambda \sin \left( \frac{p}{2} \right)}$$

## Solitonic Methods

- Pohlmeyer Map: strings on  $S^2 \rightarrow$  sine-gordon, giant magnons  $\rightarrow$  kinks. Here some generalised S-G model [Eichenherr & Honerkamp, 1979].
- Bäcklund Transformation: given vacuum and 1-soliton solution, we can generate 2-soliton solutions by solving a 1st-order DE.
- **Dressing Method** is a completely algebraic method of finding solutions: [Zakharov & Mikhailov, 1978] [Harnad, Saint Aubin, Shnider, 1984]

Writing string as group-valued  $\mathcal{F} = \Psi(0)$ , define auxiliary system for  $\Psi(x)$ . This must obey

$$\partial_{\pm} \Psi(x) = \frac{(\partial_{\pm} \mathcal{F}) \mathcal{F}^{-1}}{1 \pm x} \Psi(x)$$

Then new solutions are made from old via  $\Psi_{\text{new}}(x) = \chi(x) \Psi_{\text{old}}(x)$ , where the dressing matrix is

$$\chi(x) = 1 + \sum_i \frac{Q_i}{x - \lambda_i}$$

Controlled by poles  $\lambda_i$ , and some polarisations  $\varpi_i$  inside  $Q_i$ .

Applied to giant magnons by [Spradlin & Volovich, 2006]

Applied to  $CP^3$  to make:

- **“Big” solution:** A second generalisation of  $RP^2$   
[Hollowood & Miramontes] [Kalousios, Spradlin, Volovich] [Suzuki] 2009  
Two parameters (“dyonic”) but  $Q = 0$ : this never happens in  $S^5$ .
- **Elementary /  $CP^2$  solution:**  
[Hollowood & Miramontes, II] revisited their calculation to find this.  
Generalises ours: allows for  $p \neq \pi$ .

Also showed how to construct all other magnons  
as superpositions of these.

Can simultaneously find generalised sine-gordon solution [Miramontes, 2008]

$$\gamma = \mathcal{F}_0^{-1/2} \chi(1)^{-1} \chi(-1) \mathcal{F}_0^{1/2}$$

## Unified Dressing Construction [Hollowood & Miramontes, 2009, II]

The elementary magnon  
is generated by pole at

$$\lambda_1 = re^{ip/2}$$

(and image at  $\lambda_2 = 1/\lambda_1$ .)

This has velocity  $\cos$

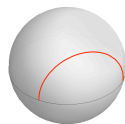
$$v_{WS} = \frac{2r}{1+r^2} \cos \frac{p}{2}$$

Two-soliton solutions with  
equal velocities can be made  
by adding a pole at

$$\lambda_3 = \frac{1}{r} e^{ip/2}$$

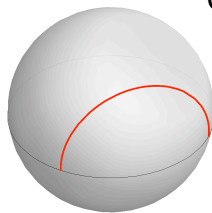
or at

$$\lambda_3 = \frac{1}{r} e^{-ip/2} = \frac{1}{\lambda_1}$$



$CP^2$  = elementary = “small”

Charges  $E_{\text{small}}, Q_{\text{small}}$



$\begin{matrix} 1, 2 \\ 3, 4 \end{matrix}$  =  $RP^3$ , with:  
 $2E_{\text{small}}, 2Q_{\text{small}}$

$\begin{matrix} 3, 4 & 1, 2 \end{matrix}$  = “Big”, with:  
 $2E_{\text{small}}, Q = 0$ .

# Finite $J$ Corrections



Finite  $J$  means finite worldsheet length.

Corrections first studied by [Arutyunov, Frolov, Zamaklar, 2006]:

$$E = \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{\rho}{2}\right) \left[ 1 - 4 \cos(2\phi) \sin^2\left(\frac{\rho}{2}\right) e^{-2\Delta/E} + o(e^{-4J/E}) \right]$$

Can also be understood as periodic solutions in  $S^2/\mathbb{Z}_n$  [Astolfi, Forini, Grignani, Semenoff] and are related to kink-train solutions of sine-gordon [Okamura & Suzuki, 2006]

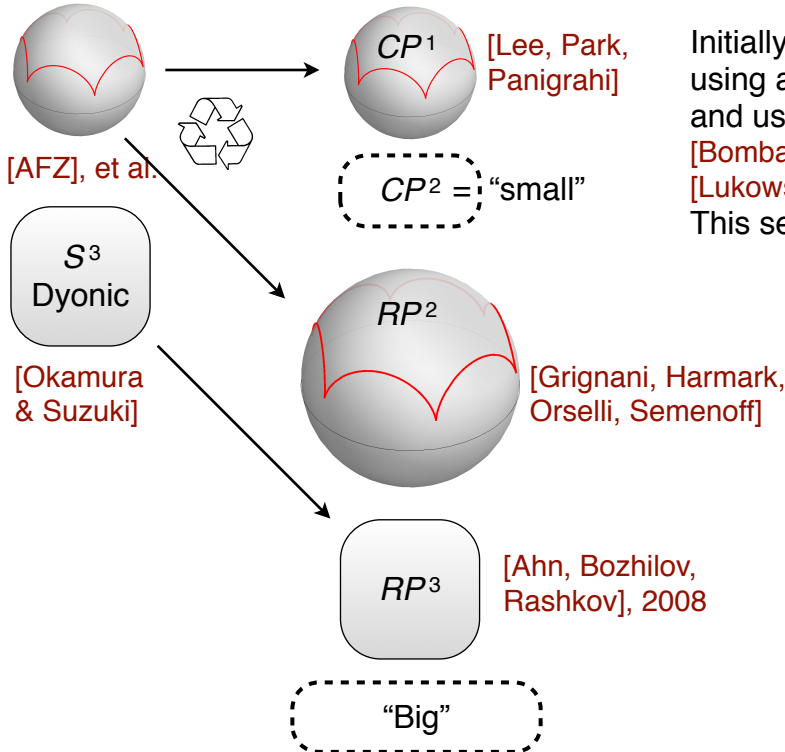
Dyonic giant magnons similarly [Hatsuda & Suzuki, 2008]

(and almost simultaneously using curves [Minahan & Ohlsson Sax]):

$$E = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{\rho}{2}\right)} - \frac{16\lambda}{\pi^2} \cos(2\phi) \frac{1}{E} \sin^4\left(\frac{\rho}{2}\right) e^{-\Delta E/2S}$$

where  $S = \frac{Q^2}{4 \sin^2(\frac{\rho}{2})} + \frac{\lambda}{4\pi^2} \sin^2\left(\frac{\rho}{2}\right) \rightarrow \frac{1}{4} E^2$  when  $Q \rightarrow 0$ .

# Finite $J$ in $CP^3$



Initially  $\delta E = 0$  for "small" using algebraic curves, and using Lüscher: [Bombardelli & Fioravanti] [Lukowski & Ohlsson Sax] This seemed strange...

## Algebraic Curves

This is an elegant re-formulation of strings in terms of Riemann surfaces.

Write strings using flat currents  $j_\mu = (\partial_\mu \mathcal{F}) \mathcal{F}^{-1}$   
and define the Lax connection

$$J(x) = \frac{1}{1-x^2} j + \frac{x}{1-x^2} * j$$

and the monodromy matrix

$$\Omega(x) = P e^{\int_\gamma d\sigma J_\sigma(x)}$$

Then  $\text{eig}(\Omega)$  is a multi-valued function of  $x$  called the algebraic curve.

We can read charges off from asymptotic behaviour:

$$\Omega(x) = 1 + \frac{1}{x} \int d\sigma j_\tau + \dots \quad \text{as } x \rightarrow \infty$$

[Mandal, Suryanarayana, Wadia] [Bena, Polchinski, Roiban] [Kazakov, Marshakov, Minahan, Zarembo] ...

- We deal with the quasi-momenta

$$\text{eig}(\Omega) = \{ e^{ip_1(x)}, e^{ip_2(x)}, \dots \}$$

- The vacuum is a set of poles at  $\pm 1$ :

$$p_1(x) = \frac{\Delta}{2g} \frac{x}{x^2 - 1}$$

- Giant magnons add to this single log cuts [Vicedo, 2006]

$$G_{\text{mag}}(x) = \frac{1}{i} \log \frac{x - X^+}{x - X^-}$$

from which we read off charges

$$\Delta - J = -ig \left( X^+ - \frac{1}{X^+} - X^- + \frac{1}{X^-} \right)$$

$$Q = -ig \left( X^+ + \frac{1}{X^+} - X^- - \frac{1}{X^-} \right)$$

and  $p = -i \log \left( \frac{X^+}{X^-} \right)$ .

## Algebraic Curves for $AdS_4 \times CP^3$

Now 10 sheets, 5 of them independent, with [Gromov & Vieira, 2008]

1. Synchronised poles:  $q_{1,2,3,4}(x) \sim \frac{\alpha x}{x^2 - 1}$  near  $x = \pm 1$

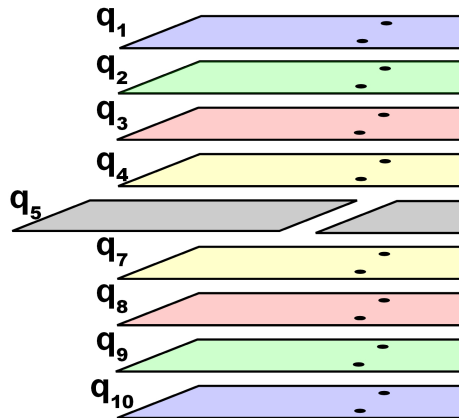
2. Inversion symmetries:  $q_1(\frac{1}{x}) = -q_2(x)$ ,  $q_3(\frac{1}{x}) = -q_4(x) + p$ ,  
 $q_5(\frac{1}{x}) = -q_5(x)$ .

(Once more  $\sum p = 0, 2\pi \dots$ )

3. Asymptotic charges: as  $x \rightarrow \infty$ ,

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} \sim \frac{1}{x} \sqrt{\frac{2}{\lambda}} \begin{pmatrix} \Delta + S \\ \Delta - S \\ \frac{1}{2}(J + Q) \\ \frac{1}{2}(J - Q) \\ J_3 \end{pmatrix}$$

4. Continuity:  $q_i(x) - q_j(x) = 2\pi n_{ij}$ ,  $x \in C_{ij}$ .



Ansatz satisfying these (with  $S = 0$ ) has three arbitrary functions

$$q_1(x) = \frac{\alpha x}{x^2 - 1}$$

$$q_2(x) = \frac{\alpha x}{x^2 - 1}$$

$$q_3(x) = \frac{\alpha x}{x^2 - 1} + G_u(0) - G_u\left(\frac{1}{x}\right) + G_v(0) - G_v\left(\frac{1}{x}\right) + G_r(x) - G_r(0) + G_r\left(\frac{1}{x}\right)$$

$$q_4(x) = \frac{\alpha x}{x^2 - 1} + G_u(x) + G_v(x) - G_r(x) + G_r(0) - G_r\left(\frac{1}{x}\right)$$

$$q_5(x) = G_u(x) - G_u(0) + G_u\left(\frac{1}{x}\right) - G_v(x) + G_v(0) - G_v\left(\frac{1}{x}\right)$$

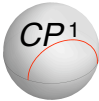
We make giant magnons by setting  $G_\gamma = G_{\text{mag}}$ , and read off charges...

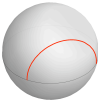
$SU(4)$  charges are

$$\begin{pmatrix} \frac{1}{2}(J + Q) \\ \frac{1}{2}(J - Q) \\ J_3 \end{pmatrix} = \begin{pmatrix} L - M_r \\ L + M_r - M_u - M_v \\ M_u - M_v \end{pmatrix}$$

Structure lines up well with that of dressing construction, although here we can still distinguish polarisations at  $Q = 1$ .

“small”

$$G_u(x) = G_{\text{mag}}(x)$$


$$G_v(x) = G_{\text{mag}}(x)$$


$$[M_U, M_V, M_R] = [1, 0, 0]$$

or  $= [0, 1, 0]$

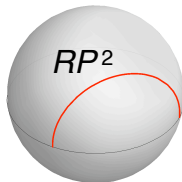
$CP^2$

$$= [Q, 0, 0] \text{ or } [0, Q, 0]$$

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“pair”

$$G_u(x) = G_v(x) = G_{\text{mag}}(x)$$



$$= [1, 1, 0]$$

$RP^3$

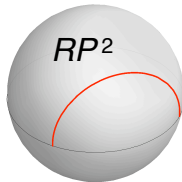
$$= [Q, Q, 0] / 2$$

[Lukowski & Ohlsson Sax]

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“big”

$$G_u(x) = G_v(x) = G_r(x) = G_{\text{mag}}(x)$$



$$= [1, 1, 1]$$

$$= [Q_u, Q_u, Q_u] / 2$$

[Shenderovich]

## Finite- $J$ magnons

Replace  $G_{\text{mag}}(x)$  with this [Minahan & Ohlsson Sax, 2008]

$$G_{\text{finite}}(x) = -2i \log \left( \frac{\sqrt{x - X^+} + \sqrt{x - Y^+}}{\sqrt{x - X^-} + \sqrt{x - Y^-}} \right)$$

where  $Y^\pm = X^\pm (1 \pm i\delta e^{\pm i\phi})$ .

- Work out asymptotic charges, expand in  $\delta$ .
- Continuity demands  $q_i - q_j = 2\pi n$  across  $\sqrt{\phantom{x}}$  branch cut, this fixes  $\delta$ .

For the  $RP^3$  / “pair” /  $(1,1,0)$  magnon,

$$\delta E = -32\lambda \cos(2\phi) \frac{1}{E} \sin^4 \left( \frac{\rho}{4} \right) e^{-\Delta E / 2S(\frac{\rho}{4})} + \mathcal{O}(\delta^3)$$

by exactly the same calculation as in  $S^5$  case.

(Here  $S(\frac{\rho}{4}) = \frac{Q^2}{16 \sin^2(\frac{\rho}{4})} + 2\lambda \sin^2 \left( \frac{\rho}{4} \right) \rightarrow \frac{1}{4} E^2$  when  $Q \rightarrow 0$ .)

For the “**small**” /  $CP^2$  /  $(1,0,0)$  magnon, [MCA, Aniceto, Ohlsson Sax, 2009]

$$\delta E = -32g^2 \cos(2\phi) \frac{Q}{E\sqrt{S(\frac{\rho}{2})}} \sin^3\left(\frac{\rho}{2}\right) e^{-\Delta E/S(\frac{\rho}{2})} + \dots$$

But does  $\delta E \rightarrow 0$  as  $Q \rightarrow 0$ ? No, there is an order-of-limits problem. By expanding in  $\delta$  we assumed  $Q/\sqrt{\lambda} \gtrsim \delta^2 \dots$

Treating this carefully, recover **AFZ** form

$$\delta E_{r=1} = -16g \cos(2\phi) \sin^3\left(\frac{\rho}{2}\right) e^{-2\Delta/\varepsilon} + \dots$$

For the “**big**” /  $(1,1,1)$  magnon,

$$\delta E = -1024g^2 \cos(2\phi) \frac{S(\frac{\rho}{4})}{EQ_u^2} \sin^6\left(\frac{\rho}{4}\right) e^{-\Delta E/S(\frac{\rho}{4})} + \dots$$

also with a delicate  $Q_u \rightarrow 0$  AFZ limit.

# Giant Magnons for SYM and ABJM Theories

(Part Two)

Michael Abbott 🇮🇳

TIFR 🇮🇳,

~~11 December 2009~~ →

19 February 2010

Describing work with Inês Aniceto 🇵🇹 and Olof Olhsson Sax 🇸🇪  
in arxiv:0811.2423 and 0903.3365,

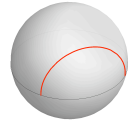
and work scooped by Hatsuda & Tanaka 🇯🇵, 0910.5315,  
about Scattering (worldsheet + spin chain)

and work in progress with Inês Aniceto 🇵🇹 and Diego Bombardelli 🇮🇹.  
on Quantum Corrections, and  $h(\lambda)$

$$\lambda \approx 0 \quad \longleftrightarrow \quad \lambda \approx \infty$$

**Review:**

Spin chain  $O = \sum_x e^{ipx} \text{Tr} \left( ZZ \underset{x}{X} ZZ \dots Z \right) \longleftrightarrow$



Giant Magnon  
[Hofman & Maldacena]

Bound states  $\dots XXXXX \dots \longleftrightarrow$

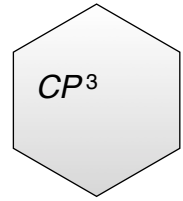


[Dorey]

$$E(p) \equiv \Delta - J = \sqrt{Q^2 + f(\lambda)^2 \sin^2 \frac{p}{2}}$$

$$f(\lambda) = \lambda$$

**ABJM case:** vacuum  $O_{vac} = \text{Tr} \left( Y^1 Y_4^\dagger \right)^J \longleftrightarrow$



Variety of sphere-like subspaces, radius  $R$  or  $R/2$

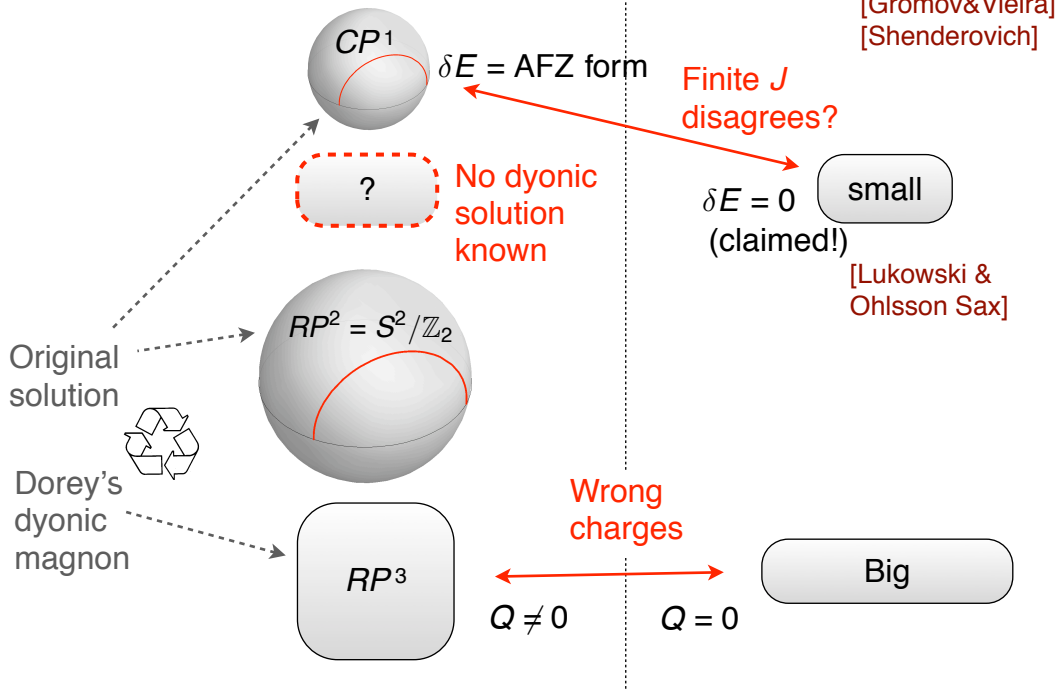
$$\Delta = \frac{J}{2} + \frac{\lambda^2}{4} \left( \sum_{i \text{ even}} \mathcal{H}_{i,i+2} + \sum_{i \text{ odd}} \mathcal{H}_{i,i+2} \right) + \dots$$

$$E = \Delta - \frac{J}{2} = \sqrt{\frac{1}{4} + 4 h^2(\lambda) \sin^2 \frac{p}{2}}$$

# Classification of Magnons, 2008

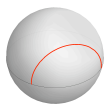
*sigma-model*

*algebraic curve*



Classification, 2009:

Elementary  
Magnon



CP<sup>2</sup>

[AAO]

*algebraic curve*

small

*Dressing constr.*

[H&M II]

1, 2

(two poles)

Composite solutions:

$RP^3$

pair of  
small

1, 2

3, 4

Big/Dressed

Big

3, 4

1, 2

[H&M, KVS, S]

(four poles)

Finite- $J$  corrections are most easily computed in the algebraic curve.

Results for  $RP^3$  like  $S^3$  (embedding) but others differ, and don't simply add.

[AAO]

# Magnon Scattering

(On the worldsheet,  $\neq$  Veneziano,  $\neq$  QCD!)

- **String side:** exact scattering solutions [Spradlin & Volovich, 2006] or use Pohlmeyer map to sine-gordon scattering solution

$$\tan \frac{\alpha}{4} = \frac{\cosh(\gamma vt)}{v \sinh(\gamma x)}$$

with time delay  $\Delta t = \frac{\sqrt{1-v^2}}{v} \log(v^2)$  (in CM frame).

- **Gauge side:** phase shift  $-i \log S$  equivalent to

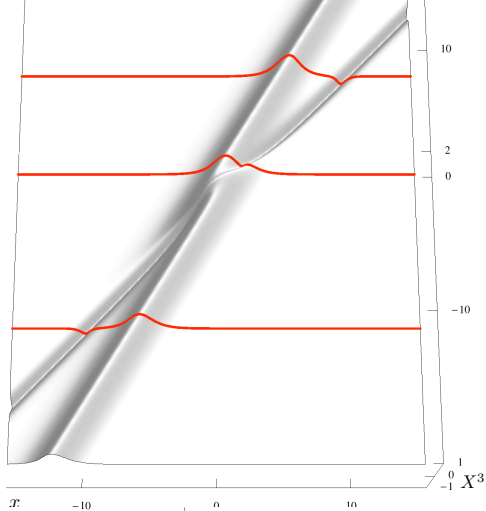
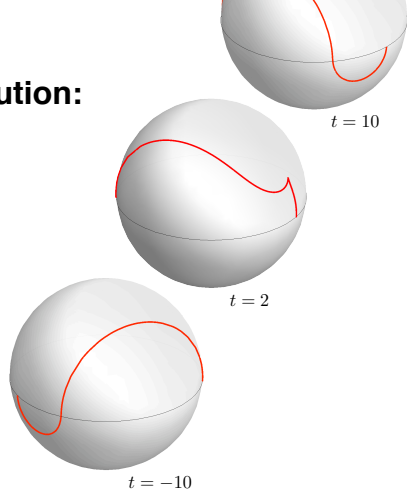
$$\Delta t = -i \frac{\partial}{\partial E} \log S$$

shown to match in first paper [Hofman & Maldacena 2006] (up to a choice of gauge).

This represents a test of the all-loop S-matrix  $S = \hat{s} \sigma$ , in fact only dressing phase  $\sigma$  contributes.

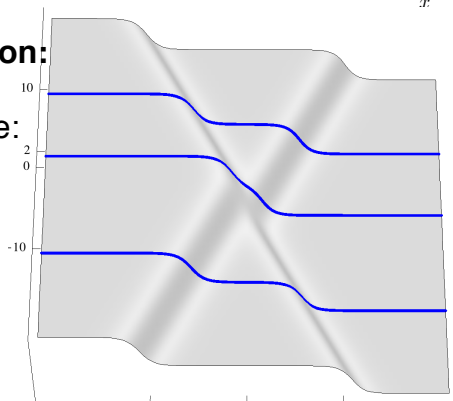


# String Solution:

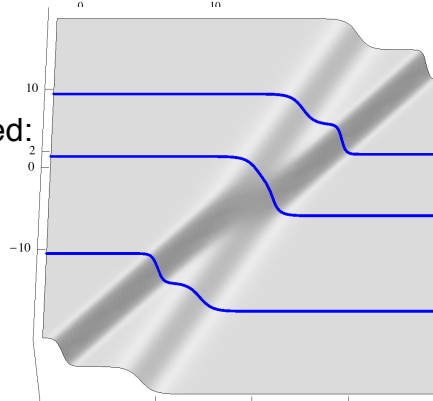


# Sine-Gordon:

CM frame:



Boosted:



## Bound-state Scattering

Extended to dyonic giant magnons by [Chen, Dorey, Okamura, 2006]:

- **String side:** use map to complex sine-gordon model

$$\Delta t = i \frac{((X^+)^2 - 1) ((X^-)^2 - 1)}{(X^+)^2 - (X^-)^2} \log \frac{(X^+ - Y^+)(X^- - Y^-)}{(X^+ - Y^-)(X^- - Y^+)}$$

- **Gauge side:** use factorised scattering

$$S(X^\pm, Y^\pm) = \prod_{i=1}^Q \prod_{j=1}^{Q'} s(x_i^\pm, y_j^\pm)$$

and  $x_i^- = x_{i-1}^+ \dots$  to simplify.

These also match (again after correcting for difference of gauge).

Here both  $\hat{s}$  and  $\sigma$  contribute.

String solutions later constructed by dressing [Kalousios, Spradlin, Volovich, 2007] showing relative orientations in  $S^5$  don't matter.

## Scattering in $CP^3$

The S-matrix proposed by [Ahn & Nepomechie, 2008] has new scalar factors

$$s^{AA} = \hat{s} \cdot \sigma \cdot \left( \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^- y^+}} \right), \quad s^{AB} = \hat{s} \cdot \sigma \cdot \left( \frac{x^- - y^+}{x^+ - y^-} \right)$$

Can reconstruct all-loop Bethe ansatz of [Gromov & Vieira, 2008] from this.

Non-dyonic case is trivial: embed old string solutions, and only  $\sigma$  matters.  
But dyonic / bound state scattering is not:

- **String side:** generalised sine-gordon solutions not known, but thanks to [Hollowood & Miramontes, 2009] we can construct such solutions using the dressing method.
- **Gauge side:** factorised scattering again...

Both sides done by [Hatsuda & Tanaka, 2009].

Different orientations lead to 4 cases, all match.

# Quantum Corrections: $\lambda < \infty$

One-loop corrections are (roughly)

$$\delta E = \sum_i (-1)^{F_i} \frac{1}{2} \hbar \omega_i \quad \left( \hbar = \frac{1}{\sqrt{\lambda}} \right)$$

Ways to work this out:

1. Expand the action, find explicit modes  $\omega_i$ , sum...
2. Add poles to algebraic curve...

or else

3. Use the Bethe ansatz, and expand...
4. Use Lüscher's formula from S-matrix...

In old  $AdS_5 \times S^5$  case resulting  $\delta E$ s are identical (at  $J = \infty$ ).

Arutyunov, Russo, Tseytlin, Park, Frolov, Tirziu, Papathanasiou, Spradlin, Gromov, Schäfer-Nameki, Vieira, Dorey, Freyhult, Beisert, Heller, Janik, Lukowski...



## One-loop Disagreement (2008)

For a spinning strings with  $AdS$  momentum  $S$ ,

$$\Delta - S = f(\lambda) \log S$$

Explicit mode calculations [McLoughlin & Roiban] [Alday, Arutyunov, Bykov] [Krishnan] gave one answer

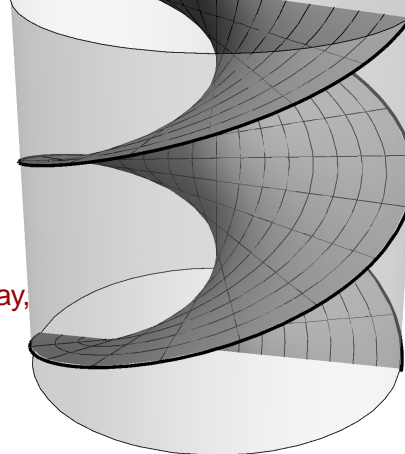
$$f(\lambda) = \sqrt{2\lambda} - 5 \frac{\log 2}{2\pi} + \dots$$

while  $sl(2)$  Bethe ansatz calculations [Gromov & Vieira] gave instead

$$f(\lambda) = 2h - 3 \frac{\log 2}{2\pi} + \dots$$

The puzzle can be resolved in two ways:

- By noting that the latter is in  $h(\lambda)$  while the former is in  $\lambda \dots$
- By altering the sum prescriptions used...



## Resolution using $h(\lambda)$ ?

The action and the algebraic curve know only about  $\lambda$ , while the Bethe ansatz (and the S-matrix) know only about  $h(\lambda)$ .

We can remove the disagreement using [McLoughlin, Roiban, Tseytlin, 2008]

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} - \frac{\log 2}{2\pi} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

This should also be visible in leading quantum correction to the magnon:

$$\begin{aligned} E_{\text{exact}} &= \sqrt{\frac{1}{4} + 4h^2 \sin^2 \frac{p}{2}} = 4h \sin \frac{p}{2} + 0 + \mathcal{O}\left(\frac{1}{h}\right) \\ &= \sqrt{2\lambda} \sin \frac{p}{2} + 2c \sin \frac{p}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) = E_{\text{class.}} + \delta E_{\text{one-loop}} + \dots \end{aligned}$$

But according to [Shenderovich, 2008],  $\delta E_{\text{one-loop}} = 0$  (for the “big” magnon).

## Resolution using $\sum_{\text{heavy,light}}$ ?

Bosonic modes of the point particle / PP-wave excitations are

- Four light modes  $\omega_n = \sqrt{\frac{1}{4} + \frac{n^2}{4\alpha^2}}$  from  $CP$  directions
- Four heavy modes  $\omega_n = \sqrt{1 + \frac{n^2}{4\alpha^2}}$  (3  $AdS$  + 1  $CP$ )

To construct modes using the curve, add poles connecting two sheets  $(i, j)$  at a point s.t.

$$q_i(x) - q_j(x) = 2\pi n$$

For the point particle:

$$x_n^{\text{light}} = \frac{\alpha}{4\pi n} + \sqrt{\frac{\alpha^2}{16\pi^2 n^2} + 1}$$
$$x_n^{\text{heavy}} = \frac{\alpha}{2\pi n} + \sqrt{\frac{\alpha^2}{4\pi^2 n^2} + 1} = x_{n/2}^{\text{light}}$$

We can make this distinction for any solution: modes  $(5, i)$  are light.

Three sum prescriptions:

1. **Old** / naïve sum [string papers M&R, AAB, K]

$$\delta E = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \sum_{ij} (-1)^{F_{ij}} \frac{1}{2} \omega_n^{ij} = \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{n=-N}^N \left( \omega_n^{\text{heavy}} + \omega_n^{\text{light}} \right)$$

2. [Gromov & Mikhaylov, 2008] proposed a **New** sum

$$\delta E = \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{n=-N}^N K_n \quad \text{where } K_n = \begin{cases} \omega_n^{\text{heavy}} + \omega_{n/2}^{\text{light}} & n \text{ even} \\ \omega_n^{\text{heavy}} & n \text{ odd} \end{cases}$$

which cuts off light modes at  $N/2$  (approx similar  $|x|$ )

3. [Bandres & Lipstein, 2009] propose instead (omiting odd heavy modes)

$$\delta E = \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{n=-N}^N \left( 2\omega_{2n}^{\text{heavy}} + \omega_n^{\text{light}} \right)$$

Using 2 / 3, disagreement with Bethe ansatz goes away, keeping  $c = 0$ .  
This is for  $AdS$  strings, we'd like to look at solutions in  $CP^3$  too.

Three sum prescriptions: (another form, and a continuum limit)

1. **Old** sum:

$$\delta E = \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{n=-N}^N (\omega_n^{\text{heavy}} + \omega_n^{\text{light}}) \rightarrow \int_{-\infty}^{\infty} dn (\omega_n^{\text{heavy}} + \omega_n^{\text{light}})$$

2. **New** sum of [G&V]:

$$\delta E = \lim_{N \rightarrow \infty} \frac{1}{2} \left( \sum_{n=-N}^N \omega_n^{\text{heavy}} + \sum_{n=-N/2}^{N/2} \omega_n^{\text{light}} \right) \rightarrow \frac{1}{2} \int_{-\infty}^{\infty} dn \left( \omega_n^{\text{heavy}} + \frac{1}{2} \omega_{n/2}^{\text{light}} \right)$$

Last mode included:  $x_N^{ij}(\text{heavy}) \approx x_{N/2}^{ij}(\text{light}) \approx 1 + \epsilon$ .

3. Alternative new sum of [B&L]:

$$\delta E = \lim_{N \rightarrow \infty} \frac{1}{2} \left( \sum_{\substack{n=-N \\ n \text{ even}}}^N 2\omega_n^{\text{heavy}} + \sum_{n=-N/2}^{N/2} \omega_n^{\text{light}} \right) \rightarrow \text{same} \int_{-\infty}^{\infty} dn$$

## Quantising Giant Magnons

To work out  $\omega_n^{ij}$  for a given polarisation, perturb quasimomenta by  $\delta q(x)$  :

- Sheets  $i, j$  contain a new pole

$$\delta q_i(x) = \frac{\alpha(y)}{x - y}, \quad \alpha(y) = \frac{1}{2g} \frac{y^2}{y^2 - 1}$$

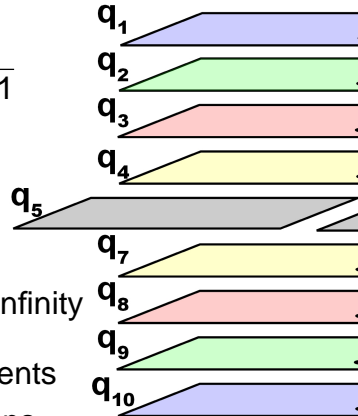
at position  $y = x_n^{ij}$  from

$$q_i(x_n^{ij}) - q_j(x_n^{ij}) = 2\pi n$$

- $\delta q$  obeys other rules from classical curve, and at infinity

$$\delta q \rightarrow \begin{cases} 0 \pm \frac{1}{2gx} + \dots & \text{for } i, j \text{ components} \\ 0 + \frac{1}{2gx} \delta \Delta + \dots & \text{in } AdS \text{ directions} \end{cases}$$

- This energy shift is  $\delta \Delta = \omega_n^{ij}$ .



## Efficient Method

[Gromov, Schäfer-Nameki, Vieira, 2008] for  $S^5$   
[Bandres & Lipstein, 2009] for  $CP^3$

- Allow arbitrary pole position  $y$ , and compute only

$$\omega^{15} = \Omega_{15}(y) \quad \text{and} \quad \omega^{45} = \Omega_{45}(y)$$

- Then inversion condition  $\delta q_3(\frac{1}{y}) = -\delta q_4(y)$  gives

$$\Omega_{35}(y) = \Omega_{45}(0) - \Omega_{45}(\frac{1}{y}) \quad \text{etc.}$$

(using unphysical poles  $|1/y| < 1$ ) and heavy modes are constructed

$$\Omega_{17}(y) = \Omega_{15}(y) + \Omega_{45}(y) \quad \text{etc.}$$

(by cancelling poles on two sheets) and we get all the  $\Omega_{ij}(y)$

- Now restore correct pole positions to write

$$\omega_n^{ij} = \Omega_{ij}(x_n^{ij})$$

An ansatz like this (written for (1, 5) polarisation)

$$\delta q_1 = \frac{\alpha(y)}{x-y} + \sum_{\pm} \frac{a_{\pm}}{x \pm 1} \qquad \delta q_2 = -\delta q_1\left(\frac{1}{x}\right)$$

$$\delta q_4 = \sum_{\pm} \frac{a_{\pm}}{x \pm 1} + M(x) \qquad \delta q_3 = -\delta q_4\left(\frac{1}{x}\right)$$

$$\delta q_5 = \frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{\frac{1}{x}-y} + \frac{\alpha(y)}{y} + M(x) - M(0) + M\left(\frac{1}{x}\right).$$

where  $M(x)$  allows cut to move

$$M(x) = -iA^+ \frac{\partial}{\partial X^+} G_{\text{mag}}(x) + iA^- \frac{\partial}{\partial X^-} G_{\text{mag}}(x) = \sum_{\pm} \frac{A^{\pm}}{x - X^{\pm}}$$

leads us to the very simple answer

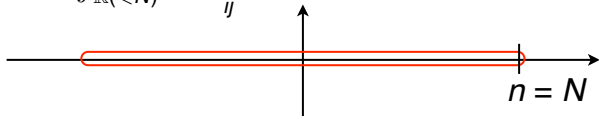
$$\Omega_{ij}(y) = \begin{cases} \Omega_{45}(y), & (i, j) \text{ light} \\ 2\Omega_{45}(y), & (i, j) \text{ heavy} \end{cases}$$

Only new feature: factor 2.

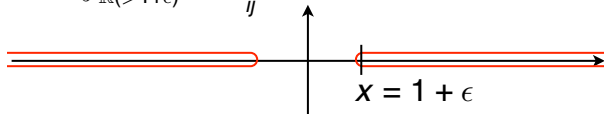
We can't find  $\omega_n = \Omega_{ij}(x_n^{ij})$  in closed form,  
but can perform these sums:

$$q_i(x_n^{ij}) - q_j(x_n^{ij}) = 2\pi n$$

$$\delta E = \lim_{N \rightarrow \infty} \frac{1}{4i} \oint_{\mathbb{R}(<N)} dn \sum_{ij} (-1)^{F_{ij}} \cot(\pi n) \Omega_{ij}(x_n^{ij})$$



$$= \lim_{\epsilon \rightarrow 0} \frac{1}{8\pi i} \oint_{\mathbb{R}(>1+\epsilon)} dx \sum_{ij} (-1)^{F_{ij}} [q'_i(x) - q'_j(x)] \cot\left(\frac{q_i(x) - q_j(x)}{2}\right) \Omega_{ij}(x)$$



$$= \lim_{\epsilon \rightarrow 0} \frac{-1}{8\pi i} \oint_{\mathbb{U}(1+\epsilon)} +$$

$$\approx \frac{-1}{4\pi i} \int_{\mathbb{U}_+} dx \sum_{ij} (-1)^{F_{ij}} [q'_i(x) - q'_j(x)] i \Omega_{ij}(x) + \text{finite-}J \text{ corrections}$$

## Results:

Giant Magnons in  $S^5$ : this gives  $\delta E = 0$ , as expected.

Elementary Magnons in  $CP^3$ :

- Using  $\sum_{\text{NEW}}$ :

$$\delta E = 0 \quad \Rightarrow \quad c = 0$$

Naïve integration in  $x$  leads you to the **new** sum.

This is how [Shenderovich] obtained  $\delta E = 0$ .

- Using  $\sum_{\text{old}}$ :

$$\delta E = -\frac{\log 2}{\pi} \sin \frac{p}{2} \quad \Rightarrow \quad c = -2 \frac{\log 2}{2\pi}$$

Here we must explicitly integrate at finite radius:

$$\lim_{\epsilon \rightarrow 0} \left[ \int_{U(2\epsilon)} \text{heavy} + \int_{U(\epsilon)} \text{light} \right]$$

This is now consistent with what was found in  $AdS$ ,  
but does not resolve the ambiguity.

## Quantum Finite- $J$ Effects:

Energy + corrections organised like this: [Gromov, Schäfer-Nameki, Vieira, 2008]

$$E = \sum_{m,n=0,1,2,\dots} a_{m,n} \left( e^{-\Delta/\sqrt{2\lambda}} \right)^m \left( e^{-2\Delta/E} \right)^n$$

- **Infinite- $J$**  piece is what we discussed:

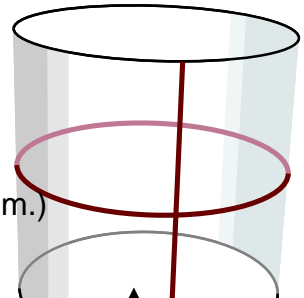
$$a_{0,0} = E_{\text{class.}} + \delta E + \dots$$

- **“F-term”**  $a_{1,0}$  is purely quantum.

Easy to calculate, simply the next term in  $\cot(q_{ij}/2) = \pm i \pm 2ie^{\pm iq_{ij}} + \dots$

$$\delta E^{F,1} = e^{-\Delta/\sqrt{2\lambda}} \sqrt{\frac{2\sqrt{2\lambda}}{\pi\Delta}} \left( \frac{\cos \frac{\rho}{2}}{1 - \sin \frac{\rho}{2}} - 1 \right)$$

Matches Lüscher results, and  $\sum_{\text{old}} = \sum_{\text{NEW}}$ .  
(Although we can rule out [B&L]’s alternative new sum.)



$$E = \sum_{m,n=0,1,2\dots} a_{m,n} \left( e^{-\Delta/\sqrt{2\lambda}} \right)^m \left( e^{-2\Delta/E} \right)^n$$

- “ $\mu$ -term”  $a_{0,1}$ :

This term contains the classical finite- $J$  corrections, so like  $a_{0,0}$ , one-loop calculation can teach us about  $c$ :

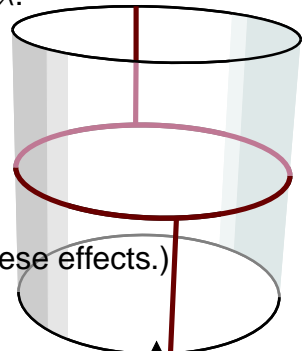
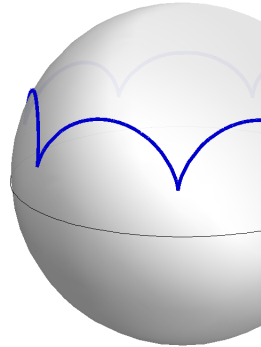
$$a_{0,1} = \sqrt{\lambda} [\text{classical}] + [\text{one-loop}] + o\left(\frac{1}{\sqrt{\lambda}}\right)$$

[Lüscher, 1986] studied finite-volume corrections to field theories in terms of the S-matrix, here that of [Ahn & Nepomechie, 2008]. This, like the Bethe ansatz and the dispersion relation, knows about  $h(\lambda)$  not  $\lambda$ :

$$a_{0,1} = h(\lambda) [\text{classical}] + [\text{one-loop}] + o\left(\frac{1}{h(\lambda)}\right)$$

[Bombardelli & Fioravanti, 2008] did the necessary subleading calculation.

(Standard “asymptotic” Bethe Ansatz does not capture these effects.)



The algebraic curve calculation involves the finite- $J$  solution, and is much more complicated. (SYM case: [Gromov, Schäfer-Nameki, Vieira, 2008, II].)

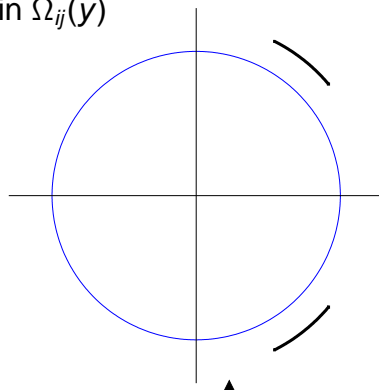
$$G_{\text{finite}}(x) = -2i \log \left( \frac{\sqrt{x - X^+} + \sqrt{x - Y^+}}{\sqrt{x - X^-} + \sqrt{x - Y^-}} \right)$$

where  $|X^+ - Y^+| = \delta \approx e^{-2\Delta/E} \ll 1$ .

Classical  $\delta E$  comes from  $o(\delta^2)$  terms in  $\Delta, J, Q, p$ .

One loop:

- Construct  $\delta q_i(x)$  (taking care about  $\sqrt{\cdot}$ ) and obtain  $\Omega_{ij}(y)$
- Some terms come from expanding  $\Omega(y)$ , but there are various extra bits...
- We want of course to do both  $\sum_{\text{NEW}}$  and  $\sum_{\text{old}}$ .



# Conclusions



We are always comparing an expansion in  $h(\lambda)$  to an expansion in  $\lambda$ . One-loop strong-coupling results:

- For *AdS* spinning strings, either  $c \neq 0$  with  $\sum_{\text{old}}$  or  $c = 0$  with  $\sum_{\text{NEW}}$  seems to be allowed.
- For giant magnons we find the same choice as for *AdS* strings, for leading corrections.
- For F-term corrections,  $\sum_{\text{NEW}} = \sum_{\text{old}}$ , as  $c$  would not affect these.

Note that this ambiguity is not OK! It alters physical energy corrections.

We hope that  $\mu$ -term calculation may shed some light.

Two-loop comparisons have been done for *AdS* strings in SYM case, e.g. [Giombi, Ricci, Roiban, Tseytlin, Vergu, 2010], but not for ABJM case.



I didn't mention new Y-system & TBA papers:

[Gromov, Kazakov, Vieira] [Arutyunov, Frolov, Suzuki] 2009

- Improvement of Bethe ansatz (all- $\lambda$ ,  $J = \infty$ ) to treat all  $J$  too?  
For both SYM and ABJM.
- On the Konishi operator (length 4) there is some disagreement with [Roiban & Tseytlin, 2009] (at  $\lambda \gg 1$ ).
- For ABJM case, I believe that this is all in terms of  $h(\lambda)$ , not  $\lambda$ .
- Tested at strong coupling only in  $AdS_5 \times S^5$  [Gromov, 2009], not in  $AdS_4 \times CP^3$ .

The End.